Proof. The distance BF is equal to a by (3.5). The lengths r/e, p/e and a/e are determined by our first definition of the ellipse (see Fig. 3.4). Then we see in Fig. 5.26 the relations $r\cos\varphi + \frac{r}{e} = \frac{p}{e}$ and $a\cos u + \frac{r}{e} = \frac{a}{e}$. These equations, when solved for r, lead to the formulas (5.48). ⌝

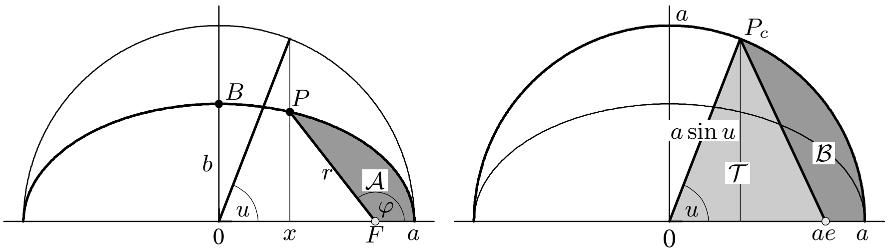

Fig. 5.27. Computation of the area $\mathcal{A}$ swept out by the radius vector

Formula for the area. The area $\mathcal{A}$ swept out by the line joining F and P (see Fig. 5.27, left) plays an important role in astronomy, as we will soon see. We stretch the ellipse into a circle (see Fig. 5.27, right) to get $\mathcal{B} = \frac{a}{b}\mathcal{A}$. Since $\mathcal{B}$ is the difference of a circular sector (of area $\frac{a^2}{2}\cdot u$) and a triangle (whose area $\mathcal{T}$ we get from Eucl. I.41 and from the fact that OF is equal to ae),[10] we obtain

$$\mathcal{B} = \frac{a^2}{2}\,(u - e\sin u) \qquad \text{and} \qquad \mathcal{A} = \frac{ab}{2}\,(u - e\sin u)\,. \tag{5.49}$$

5.10 The Great Discoveries of Kepler and Newton

> "Astronomy is older than physics. In fact, it got physics started by showing the beautiful simplicity of the motion of the stars and planets, the understanding of which was the *beginning* of physics."
> (R. Feynman, 1964, Chap. 3.4)

> " ... i libri di Apollonio, ... delle quali sole siamo bisogni nel presente trattato. [The books of Apollonius, the only ones which we require in the present treatise.]"
> (Galilei 1638, giornata quarta)

Three great works marked the emergence of modern science (see the first quotation): Kepler's *Astronomia Nova* (1609), Galilei's *Discorsi* (1638) and Newton's *Principia* (1687). The discoveries of all three works were based mainly on tools from elementary geometry (Thales, Euclid and Apollonius, see the second quotation), however in a highly ingenious way. So they fit well into our book, but don't expect easy bedtime reading here.

[10]This can be seen either from $a^2 - b^2 = a^2e^2$ (see (3.8)) and Pythagoras, or from the second formula in (5.48) with $u = 0$.

Kepler's laws.

> "... itaque futilum fuisse meum de Marte triumphum; forte fortuito incido in secantem anguli 5°.18′. quæ est mensura æquationis Opticæ maximæ. Quem cum viderem esse 100429, hic quasi e somno expergefactus, & novam lucem intuitus ... [When my triumph over Mars appeared to be futile, I fell by chance on the observation that the secant of the angle 5°18′ is 1.00429, which was the error of the measure of the maximal point. I awoke as if from sleep, & a new light broke on me.]" (J. Kepler 1609, Cap. LVI, p. 267)

Before Kepler, the knowledge in astronomy was, after thousands of years of measurements and calculations (by the Babylonian priests, Greek philosophers, Ptolemy, Copernicus' *De revolutionibus* and Tycho Brahe) as follows: The planets move around the Sun on *eccentric* circles, i.e. the Sun is not precisely at the centre of these circles. This model was quite compatible with the innumerable measurements made with unequalled precision by Tycho Brahe for all the planets known at that time, *with the exception of the planet Mars.*

After years of "pertinaci studio elaborata Pragæ", Kepler finally discovered the following laws (the first two in Kepler 1609, the last one in Kepler 1619):

Kepler 1. *Planets move on elliptic orbits with the Sun at one of the foci.*

Kepler 2. *The planets orbiting the Sun sweep out equal areas in equal time.*

Kepler 3. *The squares of the periods of revolution are proportional to the cubes of the semi-major axes.*

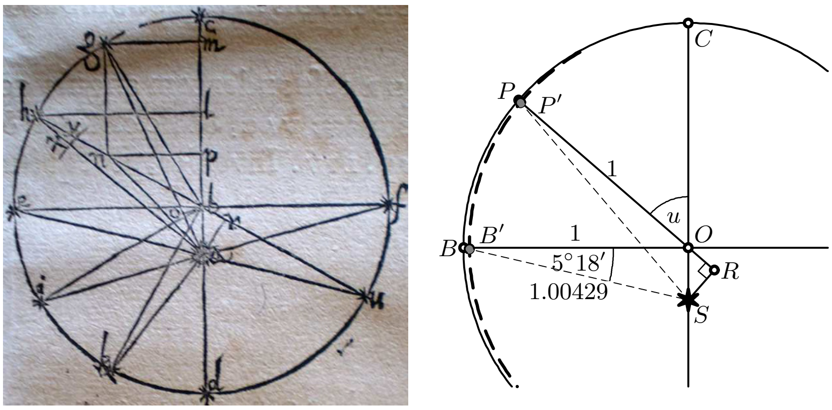

Fig. 5.28. The discovery of Kepler's first law (*Astronomia Nova*, Chap. 56); Kepler's drawing (left), modern drawing (right).

Kepler's calculations and meditations, which led to the discovery of his laws, fill hundreds of pages in his *Astronomia Nova* (1609). The decisive breakthrough occurs in Chap. 56 [11] and is explained in Fig. 5.28: The best possible circle for the orbit of Mars, which we take of radius 1, would have an eccentricity $e = OS$ such that the angle SBO is $5°18'$, where B is the point with the greatest distance from the axis SOC. But the true distance BS for Mars measured by Brahe was smaller by a factor $1/1.00429$ than the distance BS for the point on that circle. Luckily, Kepler remarked that this value is precisely $\cos 5°18'$ and "a new light broke on him" (see the quotation): we should move the point B to the point B', whose distance $B'S$ is the same as that of BO; in other words, *we have to replace the hypotenuse* (which is BS) *by the leg* (BO). Kepler tried the same recipe at other points: move the point P to the position P', such that the length $P'S$ is equal to that of the leg PR. This becomes

$$P'S = PR = 1 + e \cos u \ , \tag{5.50}$$

because the angle u, called the *eccentric anomaly*, reappears as angle SOR, so that $OR = e \cos u$. These distances (5.50), which "are confirmed by very numerous and very sure measurements" (end of Chap. 56), are precisely those of the second formula of (5.48) and thus the points *describe an ellipse*.

Newton's proof of Kepler 2. Once Kepler's laws were discovered, one wanted to understand them in the light of the foundations of mechanics, which Galilei (1638, Giornata terza) had laid down and which Newton had turned into the following two crystal clear laws:

Lex 1. Without force a body remains in uniform motion on a straight line.

Lex 2. The change of motion is proportional to the motive force impressed.

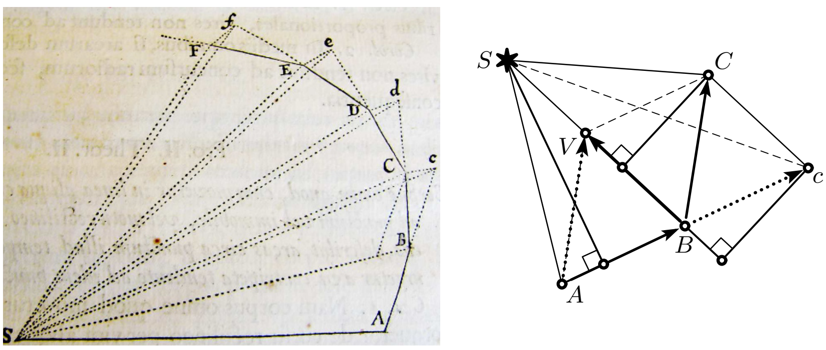

Fig. 5.29. Newton's proof for Kepler 2; reproduction from Newton's *Principia* (left); the triangles ABS, BcS and BCS having the same area (right)

[11] For more details on the first parts of the book, which culminate in the discovery of Kepler 2 (Chap. 40), we refer to Wilson (1968), Thorvaldsen (2010) and Wanner (2010).

Theorem 5.8 (Theorem 1 of the *Principia*, Engl. transl. 1729). *"The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in the same immoveable planes, and are proportional to the times in which they are described."*

Proof. We imagine a celestial body moving on an orbit $ABCDE\ldots$ under the influence of a force f acting from the Sun (see Fig. 5.29). The crucial idea is to let it advance for a certain time interval Δt *without* force from A to B (under Lex 1 uniformly on a straight line) and to compensate the missing force by one giant kick with force

$$f \cdot \Delta t \tag{5.51}$$

at the point B. Without this kick, the body would continue during the second time step Δt in uniform motion to the point c. The two triangles ABS and BcS, having the same base and the same altitude, have the same area by Eucl. I.41. Now the kick at B is in direction of the Sun, hence the velocity vector AB is transformed into a velocity AV such that BVS are aligned (by Lex 2). As a consequence, the movement for the second time interval leads from B to C in such a way that cC is parallel to BS. We conclude, again by Eucl. I.41, that the triangles BcS and BCS also have the same area.

Continuing in the same way, we find that all triangles ABS, BCS, CDS, etc., which correspond to equal time steps Δt, have equal areas. Thus Kepler's second law is proved, at least for discrete force impulses. For the case that we "now let the numbers of those triangles be augmented, and their breadth diminished *in infinitum*", Newton had prepared a "cor. 4. lem. 3." to conclude that the law will also be true in the case of a force acting "continually".[12] ⌐

This "Theorema 1" of the *Principia* did away with the first 40 chapters of Kepler's *Astronomia Nova* and its proof, more than 300 years later, has lost none of its beauty and elegance.

The discovery of the law of gravitation from Kepler 1 & 2

> "And it is the glory of Geometry that from those few principles, fetched from without, it is able to produce so many things."
> (I. Newton, from the Preface of the *Principia*, Engl. transl. 1729)

> "... one of the most dramatic moments of the *real beginnings* was when Newton suddenly understood *so* much from *so* little ..."
> (R. Feynman, lecture of March 13, 1964)

Theorem 5.9 (Prop. 11 of Newton's *Principia*). *A body P, orbiting according to Kepler* 1 *and* 2,[13] *moves under the effect of a centripetal force, directed to the centre S, satisfying the law*

[12] Today we would interpret the above procedure as a numerical method for differential equations (more precisely, the *symplectic Euler method*, cf. e.g. Hairer, Lubich and Wanner, 2006, p. 3), and rely on convergence results for such methods. The same argumentation applies to all subsequent proofs of this chapter.

[13] Not the original wording; Newton did not mention Kepler in the *Principia*.

$$f = \frac{Const}{r^2}, \qquad \textit{where } r \textit{ is the distance } SP. \tag{5.52}$$

For the *proof*, we first establish a relation between the physical force and a geometrical quantity. For this we look at Newton's drawing in his manuscript from 1684 reproduced in Fig. 5.30, left: We imagine a body moving with initial velocity in direction AB attracted by a centre of force situated far away in direction AC. This force will deviate the body during a certain time interval Δt to a curved orbit AD. If there were no initial velocity, the body would move to C, so that $ACDB$ would be a parallelogram. But the distance AC, for a fixed time interval Δt, is proportional to the force (Lex 2). We conclude that

the acting force is proportional to the distance BD between the point on the tangent and the point on the orbit. (5.53)

For this distance, denoted by RQ in the sequel, Newton discovered a nice property:

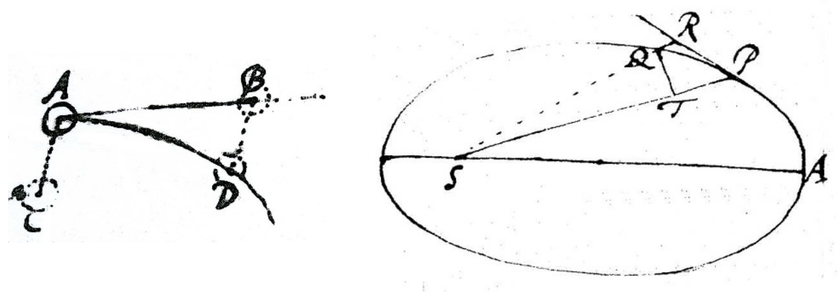

Fig. 5.30. Reproductions from Newton's autograph (1684), manuscript Cambridge Univ. Lib. Add. 3965[6]; the force acting on a moving body (left); picture for Newton's lemma (right). Reproduced by kind permission of the Syndics of Cambridge University Library

Newton's lemma. *Let APQ be an ellipse with focus S and suppose P to be the position of the planet moving towards Q, while the point R moves on the tangent with S, Q, R collinear. Let T be the orthogonal projection of Q onto PS (see Fig. 5.30, right). Then, if the distance PQ tends to zero, we have*

$$RQ \approx Const \cdot QT^2\,, \tag{5.54}$$

where the constant is independent of the position of P on the ellipse.

Proof. The proof is displayed in Fig. 5.31. We begin by collecting what we know from Apollonius (see Chap. 3): we know that the tangent PR is parallel to the diameter DCK, conjugate to GCP (Apoll. II.6). We denote the lengths of these diameters by $2d$ and $2c$ respectively. Through H, the other focus, and Q we draw parallels to DK which yield the points I and X on SP and V on

CP.[14] We further know that the normal PF of length h is the angle bisector of SPH (Apoll. III.48), i.e. the triangle IPH is isosceles, hence $IP = PH$ (Eucl. I.6). We next have $SE = EI$ by Thales since $SC = CH$ (Apoll. III.45). Therefore, since $SE + EI + IP + PH = 2a$ (Apoll. III.52), we obtain our first interesting result,

$$EP = EI + IP = a\,. \tag{5.55}$$

The *key idea* of the proof is now the following one: if our ellipse were a circle, we would know by Eucl. III.35 (or Eucl. II.14) that $GV \cdot VP = QV^2$. But in the case of the ellipse, we have to divide these values by the lengths of the corresponding conjugate diameters and obtain

$$\frac{GV \cdot VP}{c^2} = \frac{QV^2}{d^2} \qquad \text{i.e.} \qquad (3): \quad VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2}\,.$$

To complete the proof, we have to express VP in terms of RQ and QV in terms of QT. Note that the triangle XVP is similar to ECP and QTX is similar to PFE (orthogonal angles), whence by (5.55)

$$(2): \quad XP = VP \cdot \frac{a}{c}\,, \qquad (6): \quad QX = QT \cdot \frac{a}{h}\,.$$

In order to make more progress, we now leave the path of exemplary Greek rigour and suppose PQ very (infinitely) small, i.e. we identify

$$(1): \quad RQ \approx XP\,, \qquad (4): \quad GV \approx GP = 2c\,, \qquad (5): \quad QV \approx QX\,.$$

A simple calculation now gives by using, in this order, (1), (2), (3), (4), (5), (6),

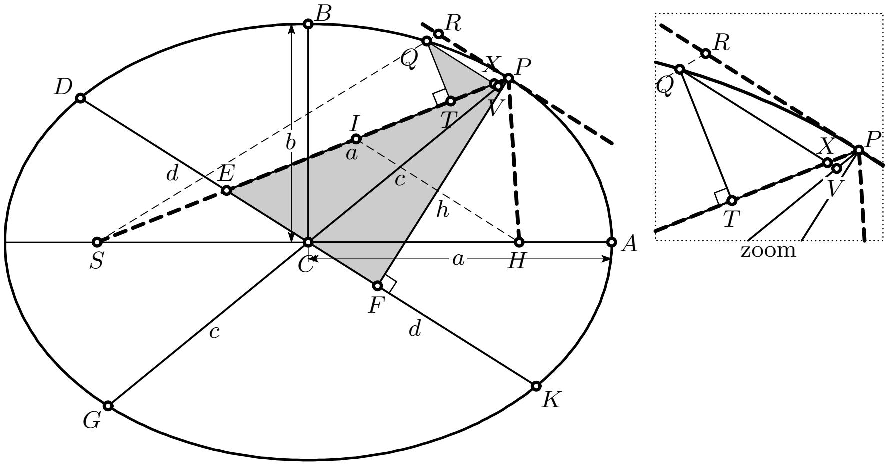

Fig. 5.31. Newton's proof of his lemma

[14] All upper case letters of this proof are the original ones of Newton, but not the lower case letters a, b, c, d and h, which we use to simplify the formulas.

$$RQ \approx \frac{a^3}{2h^2d^2} \cdot QT^2 .$$

From Apoll. VII.31 (Exercise 2 on page 73) we finally have $hd = ab$ ($\frac{1}{4}$ area of the circumscribed parallelogram), which turns the above formula into

$$RQ \approx \frac{a}{2b^2} \cdot QT^2 \tag{5.56}$$

where, as stated, the constant[15] is independent of the position of P. ⌟

Proof of Theorem 5.9. The main theorem is finally obtained by combining the above three results:

(a) The force f is proportional to RQ (equation (5.53));

(b) RQ is proportional to QT^2 (Newton's lemma (5.54));

(c) QT is inversely proportional to SP, because $\frac{QT \cdot SP}{2}$ (the area of the triangle SPQ) is constant (Δt fixed, Kepler 2);

hence f is inversely proportional to SP^2. ⊐

R. Feynman and the reciprocal problem

> "Pour voir présentement que cette courbe $ABC\ldots$ est toûjours une Section Conique, ainsi que Mr. Newton l'a supposé, *pag. 55. Coroll. I.* sans le démontrer; il y faut bien plus d'adresse. [To see now that this curve $ABC\ldots$ is always a conic section, as Mr. Newton has assumed without proof on *p. 55, Coroll. I*, requires considerably more ability.]" (Joh. Bernoulli, 1710)
>
> "... no calculus required, no differential equations, no conservation laws, no dynamics, no angular momentum, no constants of integration. This is Feynman at his best: reducing something seemingly big, complicated, and difficult to something small, simple, and easy." (B. Beckman, 2006)

The reciprocal result, that a body orbiting under the influence of a central force obeying the inverse-square law always follows an elliptic, parabolic or hyperbolic arc, was much harder to prove. Joh. Bernoulli, who gave a proof for the problem in 1710 using differential calculus, stated proudly that answering this question "requires considerably more ability" (see the quotation). A *geometric* explanation, as elegant as the proofs above, had to wait for another three centuries and was presented by R. Feynman in his lecture of March 13, 1964 at Caltech (see Feynman, Goodstein and Goodstein 1996, also Beckman 2006).[16]

[15] Newton remarked that this constant is the reciprocal of the latus rectum.

[16] The authors are grateful to Christian Aebi and Bernard Gisin, Geneva, for valuable references to the literature.

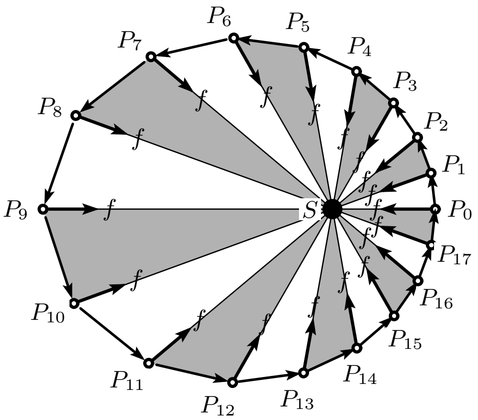

Fig. 5.32. Feynman's variant with equal angles instead of equal time steps

Equal angles instead of equal time steps. We suppose that we have a force acting according to the inverse-square law. As we observed in (5.51) and (5.52), the force impulses, for constant time steps, decrease like $\frac{1}{r^2}$ with increasing r. We now choose *equal angles* at the Sun. By Eucl. VI.19, the areas of the triangles SP_iP_{i+1} are proportional to r^2. Therefore, by Kepler 2, the time steps Δt (which multiply the force) are proportional to r^2 as well and

$$\text{\textit{the force impulses will all have the same length.}} \quad \text{Moreover, \textit{their directions form a regular star.}} \tag{5.57}$$

The situation is summarised in Fig. 5.32.

The hodograph. We now draw the *velocities* as points in a space with origin O (see Fig. 5.33, left). The velocity $\dot{P}_0$ at the perihelion P_0 is fastest and directed upwards. Then the impulses f push the velocities $\dot{P}_1$, $\dot{P}_2, \ldots$ first to the left, then downwards, until at the aphelion (here P_9) the velocity is slowest and directed exactly downwards. All impulses f have, by (5.57), constant length and their directions increase regularly by the same amount $\Delta\varphi$. We therefore get a regular n-gon and, for $\Delta\varphi \to 0$, we obtain the surprising result:

$$\text{\textit{The velocity} } \dot{P} \text{ \textit{of a planet orbiting under the effect of a central force inversely proportional to} } r^2 \text{ \textit{describes a circle.}} \tag{5.58}$$

The centre C of the circle is not at the origin O, except for circular motion with constant speed. If the origin O were on or outside the circle, we would have parabolic or hyperbolic motion.

It is interesting that such an elegant result escaped the attention of Euler, Lagrange and Laplace. Only in the work of Hamilton did the velocities (momenta) acquire the same importance as the positions.

Conclusion. Now comes the most difficult step (Feynman: "I took a long time to find that"). We have to find a connection between the orbit in Fig. 5.32

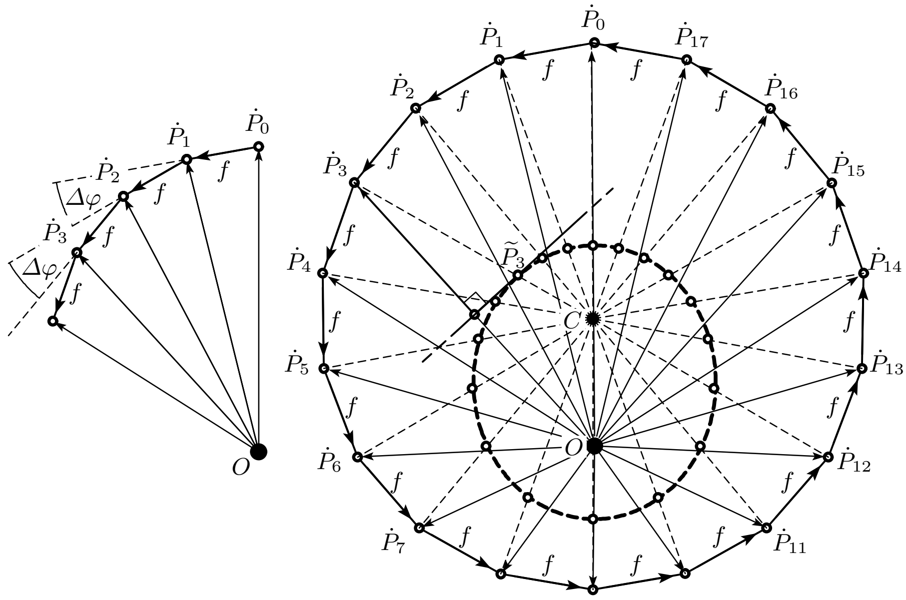

Fig. 5.33. The hodograph of Kepler motion, producing a circle (left); synthesis of the two pictures (right)

and the hodograph in Fig. 5.33. After some time for thought, we draw in the second picture the curve of points P having the same distance from O as from the circle and obtain Fig. 5.33 (right). We know from Chap. 3 (see in particular Fig. 12.8 (b)) that this curve is an ellipse with O and C as foci. We denote by $\widetilde{P}_i$ the points of this ellipse situated on the rays $C\dot{P}_i$. These points are located under the same angles with respect to C as are the corresponding points P_i with respect to S in Figure 5.32.

We next consider Fig. 3.4 (right): the tangent at P is orthogonal to FB (in the notation of that figure). Applied to Fig. 5.33, this means that the tangent at $\widetilde{P}_i$ to the ellipse is *orthogonal* to $O\dot{P}_i$. On the other hand, the tangent at P_i to the orbit in Fig. 5.32 (left) is *parallel* to $O\dot{P}_i$. We conclude that *the two ovals are identical, just rotated by* 90° *and, perhaps, scaled differently.* Since we know that the "oval" in Fig. 5.33 is an ellipse, with C as focus, we have that the orbit in Fig. 5.32 is also an ellipse with S as focus. ∟

This was "Feynman at his unique best" (see the quotation); later D.L. and J.R. Goodstein discovered that precisely the same proof had been published in 1877 by another great physicist, James Clerk Maxwell.

> "It is not easy to use the *geometric* method to discover things, it is very difficult, but the elegance of the demonstrations after the discoveries are made, is really very great. The power of the *analytic* method is that it is much easier to discover and to prove things, but not in any degree of elegance. There is a lot of dirty

paper with x-es and y-s and crossed out cancellations and so on ... (laughter)."

(R. Feynman, lecture of March 13, 1964, 35th minute)

This "dirty paper with x-es and y-s" leads us to the next chapters ...

5.11 Exercises

1. Prove, for the circular quadrilateral with sides a, b, c, d of Ptolemy's Lemma 5.1 and Fig. 5.4, the formulas

$$\delta_1 : \delta_2 = (ab + cd) : (ad + bc)\,, \quad \delta_1^2 = (ac + bd)(ab + cd) : (ad + bc) \qquad (5.59)$$

which can be found in Förstemann (1835).

2. Multiply the values of $\cos\alpha$ for $\alpha = 0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \frac{6\pi}{6}$ by 6 and design a simple rule for French fisherman to find the tidal height as the sea level falls, hour per hour, during approximately 6 hours from high water to low water.

3. (Exercise suggested by P. Henry (2009)) Reconstruct Viète's proof of the addition formulas (5.6) — which in Viète were not "formulas", but half a page of Latin text — by supposing $BC = \sin\alpha$, $AC = \cos\alpha$, $BD = \sin\beta$, $AD = \cos\beta$ to be known (see Fig. 5.34) and by computing, using Thales, Pythagoras and Eucl. III.20, $BE = \sin(\alpha+\beta)$ and $AE = \cos(\alpha+\beta)$.

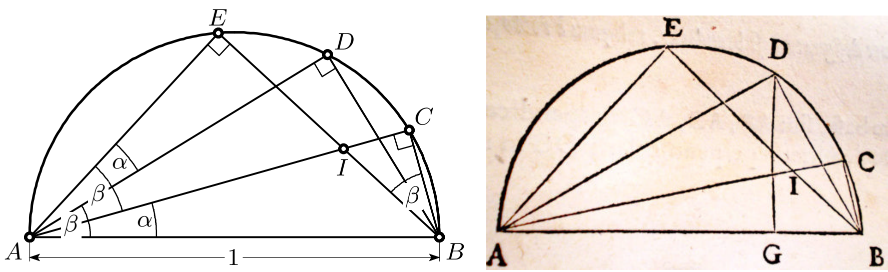

Fig. 5.34. Proof of Viète; right: illustration from van Schooten's edition 1646

4. Verify the values of sine, cosine and tangent given in Table 5.2.

5. Consider an arbitrary triangle with sides a, b, c. Prove the following beautiful expressions for the half-angles:

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}\,, \qquad \cos\frac{\alpha}{2} = \sqrt{\frac{(s-a)s}{bc}}\,,$$
$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{(s-a)s}}\,, \qquad \text{where} \quad s = \frac{a+b+c}{2} \qquad (5.60)$$